Published by Institute of Physics Publishing for SISSA

Received: *March 2, 2006*
Revised: *March 29, 2006*
Accepted: *April 18, 2006*
Published: *May 3, 2006*

# Observations on the Darboux coordinates for rigid special geometry

**Sergio Ferrara**

*Physics Department, Theory Unit, CERN*
*CH-1211 Geneva 23, Switzerland, and*
*INFN, Laboratori Nazionali di Frascati*
*Via Enrico Fermi 40, I-00044 Frascati, Italy*
*E-mail:* Sergio.Ferrara@cern.ch

**Óscar Maciá**

*Departament de Física Teòrica, Universitat de València and IFIC*
*C/Dr. Moliner, 50, E-46100 Burjassot (València), Spain*
*E-mail:* Oscar.Macia@ific.uv.es

Abstract: We exploit some relations which exist when (rigid) special geometry is formulated in real symplectic special coordinates $P^I = (p^\Lambda, q_\Lambda)$, $I = 1, \ldots, 2n$. The central role of the real $2n \times 2n$ matrix $M(\Re\mathcal{F}, \Im\mathcal{F})$, where $\mathcal{F} = \partial_\Lambda \partial_\Sigma F$ and $F$ is the holomorphic prepotential, is elucidated in the real formalism. The property $M\Omega M = \Omega$, where $\Omega$ is the invariant symplectic form, is used to prove several identities in the Darboux formulation. In this setting the matrix $M$ coincides with the (negative of the) Hessian matrix $H(S) = \frac{\partial^2 S}{\partial P^I \partial P^J}$ of a certain hamiltonian real function $S(P)$, which also provides the metric of the special Kähler manifold. When $S(P) = S(U+\bar{U})$ is regarded as a "Kähler potential" of a complex manifold with coordinates $U^I = \frac{1}{2}(P^I + iZ^I)$, it provides a Kähler metric of a hyperkähler manifold, which describes the hypermultiplet geometry obtained by $c$-map from the original $n$-dimensional special Kähler structure.

Keywords: Black Holes, Supergravity Models, Extended Supersymmetry, Differential and Algebraic Geometry.





JHEP05(2006)008

## Contents



## 1. Introduction

Special geometry [1 − 3] plays an important role in the description of the moduli space of Calabi-Yau compactifications for supergravity effective actions down to $D = 4$ dimensions [4] and also for more general compactifications when fluxes [5, 6] of different nature are turned on.

More interestingly, special geometry has served as a basis to study the so-called "attractor mechanism" [7 − 10] for black hole backgrounds preserving at most four supercharges.

In the rigid case [11 − 14], special geometry was an important tool for the Seiberg-Witten [15] analysis of non-perturbative properties of $N = 2$ super Yang-Mills theories. An important ingredient in special geometry is the existence of a flat symplectic bundle with structure group $\mathrm{Sp}(2n, \mathbb{R})$, ($\mathrm{Sp}(2n + 2, \mathbb{R})$ in the local case) [3, 16, 17, 10] over the special Kähler manifold of complex dimension $n$, $n$ being the number of vector multiplets in the theory.

It is the aim of the present work to elucidate some properties of such a rich structure when Darboux real special symplectic sections are adopted [18, 16, 17, 19] for the description of underlying mathematical structure. This description is particularly suitable when background charges are introduced which are related to fluxes of vector field-strength 2-forms over the space-time manifold. Such a description has recently been used to simplify the entropy area formula [20] for extremal black holes and its relation to superstring theory [21]. We will limit ourselves to giving general results for the case of rigid special geometry but many of these results can be extended to the local case which will be described elsewhere.



JHEP05(2006)008

We consider (rigid) special geometry in real special (Darboux) coordinates $P^I = (p^\Lambda, q_\Lambda)$, $\Lambda = 1, \ldots, n$ with the Kähler 2-form

$$\omega = i dq_\Lambda \wedge dp^\Lambda = \frac{i}{2} dP \wedge \Omega dP \tag{1.1}$$

where

$$\Omega = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$$

is the symplectic invariant form.

The special geometry data turn out to be encoded in a real "Hamiltonian" function $S(p,q)$, originaly introduced by Cecotti et al. [18] and by Freed [16], which is the Legendre transform of the imaginary part of the holomophic prepotential $F(X^\Lambda)$ of special geometry [2, 3, 22 – 24]. The holomorphic symplectic sections

$$V = (X^\Lambda, F_\Lambda) \tag{1.2}$$

are related to the real variables as follows:

$$\Re X^\Lambda = p^\Lambda \qquad \Im X^\Lambda = \phi^\Lambda = \frac{\partial S(p,q)}{\partial q_\Lambda} \tag{1.3}$$

$$\Re F_\Lambda = q_\Lambda \qquad \Im F_\Lambda = \psi_\Lambda = -\frac{\partial S(p,q)}{\partial p^\Lambda} \tag{1.4}$$

If we encode the imaginary parts in the symplectic real vector $I^I = (\phi^\Lambda, \psi_\Lambda)$, it turns out that

$$I^I = \Omega'^{IJ} \frac{\partial S}{\partial P^J} \tag{1.5}$$

with

$$\Omega'^{IJ} = -\Omega_{IJ} = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$$

In this note we will show the special property played by the $2n \times 2n$ real symmetric positive-definite matrix [23, 8]

$$M(\Re\mathcal{F}, \Im\mathcal{F}) = \begin{pmatrix} \Im\mathcal{F} + \Re\mathcal{F}\Im\mathcal{F}^{-1}\Re\mathcal{F} & -\Re\mathcal{F}\Im\mathcal{F}^{-1} \\ -\Im\mathcal{F}^{-1}\Re\mathcal{F} & \Im\mathcal{F}^{-1} \end{pmatrix} \tag{1.6}$$

which in the real formulation turns out to be related to the Hessian matrix

$$H_{IJ}(S) = \frac{\partial^2 S}{\partial P^I \partial P^J} \tag{1.7}$$

namely

$$M(p,q) = -H(S) \tag{1.8}$$

Note that $M$ is positive as a consequence of the fact that $\Im F$ is positive [23, 8] which is required by the positivity of the metric (see eq. (2.22)). The matrix $M$ is known to play a special role when a background (symplectic) charge vector $Q = (m^\Lambda, e_\Lambda)$ is introduced and a "central charge" holomorphic function

$$Z = \langle Q, V\rangle = Q^T \Omega V = X^\Lambda e_\Lambda - m^\Lambda F_\Lambda \tag{1.9}$$

is defined.

In rigid special geometry[1] the following identity holds (for the local case, see later):

$$Q + i\Omega M Q = -2ig^{\Lambda\bar{\Sigma}} \partial_\Lambda V \bar{\partial}_\Sigma \bar{Z} \tag{1.10}$$

Multiplying by $Q^T\Omega$ on the left we get the "central charge" potential function

$$\frac{1}{2} Q^T M Q = g^{\Lambda\bar{\Sigma}} \partial_\Lambda Z \bar{\partial}_\Sigma \bar{Z} \tag{1.11}$$

where $g^{\Lambda\bar{\Sigma}}$ is the inverse of the metric defined below.

The Hessian (1.7) gives also the metric of the original Kähler manifold

$$g_{\Lambda\bar{\Sigma}} dz^\Lambda \otimes d\bar{z}^\Sigma = \frac{\partial^2 K}{\partial X^\Lambda \partial \bar{X}^\Sigma} dX^\Lambda \otimes d\bar{X}^\Sigma \tag{1.12}$$

where special coordinates $z^\Lambda = X^\Lambda$ have been adopted and

$$K = -i(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda) = -i\langle V, \bar{V}\rangle$$

namely

$$g_{\Lambda\bar{\Sigma}} dz^\Lambda \otimes d\bar{z}^\Sigma = -2H_{IJ} dP^I \otimes dP^J \tag{1.13}$$

Another interesting observation is related to the $c$-map hypermultiplet geometry as defined in references [18, 26].

By adopting real symplectic coordinates $Z^I = (\zeta^\Lambda, \tilde{\zeta}_\Lambda)$ for the (other half) hypermultiplet coordinates, the hyperkähler metric has the form [18, 26, 27]

$$g_{\Lambda\bar{\Sigma}} dz^\Lambda \otimes d\bar{z}^\Sigma + 2M_{IJ}(z, \bar{z}) dZ^I \otimes dZ^J \tag{1.14}$$

Adopting Darboux coordinates for the Kähler manifold $\mathcal{M}$, and because of (1.8) and (1.13), we then have

$$-2\frac{\partial^2 S}{\partial P^I \partial P^J}(dP^I \otimes dP^J + dZ^I \otimes dZ^J) \tag{1.15}$$

By complexifying the Darboux coordinates as $U^I = \frac{1}{2}(P^I + iZ^I)$ we see that (1.14) is a Kähler metric with Kähler potential [17]

$$K(U, \bar{U}) = -8S(U + \bar{U}) \tag{1.16}$$

Note that, as expected from the results of [26], the metric (1.14) has $2n$ real isometries

$$U^I \longmapsto U^I + ia^I \tag{1.17}$$

[1] See ref. [25] for an intrinsic definition.



JHEP05(2006)008

In the "complex" formulation of Cecotti et al. [18], where the (second half of the) hypermultiplet coordinates were denoted by $W_\Lambda$, the same isometries were acting as

$$W_\Lambda \longmapsto W_\Lambda + i\alpha_\Lambda \tag{1.18}$$
$$W_\Lambda \longmapsto W_\Lambda + (\Im F)_{\Lambda\Sigma}\beta^\Sigma \tag{1.19}$$

and the Kähler potential was [18]

$$N(X^\Lambda, W_\Lambda) = K(X,\bar{X}) + (\Im\mathcal{F}^{-1})^{\Lambda\Sigma}(W_\Lambda + \bar{W}_\Lambda)(W_\Sigma + \bar{W}_\Sigma) \tag{1.20}$$

In the local case a particular choice of real coordinates has recently been used [20, 19] in conjunction with the modification of the black hole entropy formula due to higher curvature corrections and also in relating the entropy area formula with the topological string partition function [21].

The paper is organized as follows. In section 2 we give an explicit derivation of the special Kähler metric in the Darboux coordinates [16, 17] and, by using the properties of the $M$ matrix we arrive at equations (1.7), (1.13), (1.12). In section 3 we discuss real special coordinates in connection with the central charge function. We finally comment on some central charge relations and some metric differential identities for local special geometry.



## 2. Complex and real special coordinates

### 2.1 Rigid real sections and the functional $S(p, q)$

Let $(X^\Lambda, F_\Lambda)$ be the rigid holomorphic symplectic sections depending on the holomorphic coordinates $z^\Lambda$ [24]

$$X^\Lambda = X^\Lambda(z) \qquad F_\Lambda = F_\Lambda(z) \tag{2.1}$$

Under some general assumptions we can take $X^\Lambda$ as the holomorphic coordinates $X^\Lambda = z^\Lambda$ such that

$$F_\Lambda(X) = \frac{\partial F(X)}{\partial X^\Lambda} \tag{2.2}$$

for a suitable function $F(X)$.

The holomorphicity condition reads

$$\bar{\partial}_\Sigma X^\Lambda = 0 \qquad\qquad \bar{\partial}_\Sigma F_\Lambda = 0 \tag{2.3}$$

Decomposing $(X^\Lambda, F_\Lambda)$ in terms of real and imaginary parts we can write the holomorphic symplectic sections in terms of real variables

$$X^\Lambda = p^\Lambda + i\phi^\Lambda \qquad F_\Lambda = q_\Lambda + i\psi_\Lambda \tag{2.4}$$

Define the function $L = \Im F > 0$, the imaginary part of $F(X)$. Then it can be proved that $(q_\Lambda, \phi^\Lambda)$ and $(p^\Lambda, \psi_\Lambda)$ are pairs of conjugate variables for $L$

$$q_\Lambda = \frac{\partial L}{\partial \phi^\Lambda} \tag{2.5}$$

$$\psi_\Lambda = \frac{\partial L}{\partial p^\Lambda} \tag{2.6}$$

We perform a Legendre transform on $L$ of the form

$$S(p,q) = q_\Lambda \cdot \phi^\Lambda(p,q) - L(p,\phi(p,q)) \tag{2.7}$$

where we have to invert equation (2.5) to write $\phi = \phi(p,q)$. Then, the next set of equations follows:

$$\phi^\Lambda = \frac{\partial S}{\partial q_\Lambda} \tag{2.8}$$

$$-\psi_\Lambda = \frac{\partial S}{\partial p^\Lambda} \tag{2.9}$$

Our change of coordinates $(p^\Lambda, \phi^\Lambda) \mapsto (p^\Sigma, q_\Sigma)$ is of the form

$$p^\Sigma = \delta^\Sigma_\Lambda p^\Lambda \qquad q_\Sigma = q_\Sigma(p^\Lambda, \phi^\Lambda) \tag{2.10}$$

and, accordingly, the inverse change of coordinates $(p^\Sigma, q_\Sigma) \mapsto (p^\Lambda, \phi^\Lambda)$ is

$$p^\Lambda = \delta^\Lambda_\Sigma p^\Sigma \qquad \phi^\Lambda = \phi^\Lambda(p^\Sigma, q_\Sigma) \tag{2.11}$$

In the following we use the notation

$$\left.\frac{\partial f(x,y)}{\partial x}\right|_y = \frac{\partial f}{\partial x} \qquad\qquad \left.\frac{\partial f(x,y)}{\partial y}\right|_x = \frac{\partial f}{\partial y}$$

where the role of $f(x,y)$ will be played by $q_\Sigma(p,\phi)$, $\phi^\Lambda(p,q)$ as defined in eqs. (2.10), (2.11).

Let $\mathbb{J}$ be the Jacobian matrix. It follows from $\mathbb{J}(\mathbb{J})^{-1} = \mathbf{1} = (\mathbb{J})^{-1}\mathbb{J}$ that

$$\frac{\partial q_\Sigma}{\partial \phi^\Lambda}\frac{\partial \phi^\Lambda}{\partial q_\Gamma} = \delta^\Sigma_\Gamma = \frac{\partial q_\Sigma}{\partial \phi^\Lambda}\frac{\partial^2 S}{\partial q_\Lambda \partial q_\Gamma} \tag{2.12}$$

i.e.

$$\frac{\partial q_\Sigma}{\partial \phi^\Lambda} = \left(\frac{\partial^2 S}{\partial q_\Lambda \partial q_\Sigma}\right)^{-1} = \frac{\partial^2 L}{\partial \phi^\Sigma \partial \phi^\Lambda} \tag{2.13}$$

and we also have

$$\frac{\partial q_\Sigma}{\partial p^\Lambda}\delta^\Lambda_\Gamma + \frac{\partial q_\Sigma}{\partial \phi^\Lambda}\frac{\partial \phi^\Lambda}{\partial p^\Gamma} = \frac{\partial q_\Sigma}{\partial p^\Lambda}\delta^\Lambda_\Gamma + \left(\frac{\partial^2 S}{\partial q_\Sigma \partial q_\Lambda}\right)^{-1}\frac{\partial^2 S}{\partial q_\Lambda \partial p^\Gamma} = 0 \tag{2.14}$$

In fact, equations (2.12) to (2.14) follow not only from the Jacobian but also from the holomorphicity conditions (2.3). These also imply

$$\frac{\partial q_\Sigma}{\partial \phi^\Lambda} = -\frac{\partial \psi_\Sigma}{\partial p^\Lambda} - \frac{\partial q_\Gamma}{\partial p^\Lambda}\frac{\partial \psi_\Sigma}{\partial q_\Gamma} \tag{2.15}$$

where $\psi = \psi(p,q)$ and its partial derivatives are computed following the notation explained before. The analyticity of $L = \Im F$, for $F$ holomorphic, also imply

$$\frac{\partial^2 L}{\partial \phi^\Lambda \partial \phi^\Sigma} + \frac{\partial^2 L}{\partial p^\Lambda \partial p^\Sigma} = 0 \qquad\qquad \frac{\partial^2 L}{\partial \phi^\Lambda \partial p^\Sigma} = \frac{\partial^2 L}{\partial p^\Lambda \partial \phi^\Sigma}$$

In terms of $S$ eq. (2.15) can be written as

$$\left(\frac{\partial^2 S}{\partial q_\Sigma \partial q_\Lambda}\right)^{-1} = \frac{\partial^2 S}{\partial p^\Sigma \partial p^\Lambda} + \frac{\partial q_\Gamma}{\partial p^\Lambda}\frac{\partial^2 S}{\partial q_\Gamma \partial p^\Sigma} \tag{2.16}$$

These relations will be useful when simplify the metric in the next section.

## 2.2 Kähler potential and metric

In this section we show that the Kähler potential $K$ can be expressed directly in terms of $S(p,q)$. Then, the metric can also be expressed directly in terms of $S(p,q)$, since

$$g = g_{\Lambda\bar{\Sigma}} dX^\Lambda \otimes d\bar{X}^\Sigma \tag{2.17}$$

with

$$g_{\Lambda\bar{\Sigma}} = \frac{\partial^2 K}{\partial X^\Lambda \partial \bar{X}^\Sigma} \tag{2.18}$$

In the rigid case the Kähler potential is

$$K = i(X^\Lambda \bar{F}_\Lambda - \bar{X}^\Lambda F_\Lambda) \tag{2.19}$$

The Kähler form then is

$$\omega = -\frac{1}{4}\partial_\Lambda \bar{\partial}_\Sigma K \; dX^\Lambda \wedge d\bar{X}^\Sigma = -\frac{1}{2}\Im F_{\Lambda\Sigma} \; dX^\Lambda \wedge d\bar{X}^\Sigma \tag{2.20}$$

so that

$$\omega = i dq_\Lambda \wedge dp^\Lambda \tag{2.21}$$

is a symplectic form. The metric will be

$$g_{\Lambda\bar{\Sigma}} = 2\Im F_{\Lambda\Sigma} > 0 \tag{2.22}$$

Using equations (2.12) to (2.16), one arrives at

$$g_{\Lambda\bar{\Sigma}} = -2\left(\frac{\partial^2 S}{\partial p^\Lambda \partial p^\Sigma} + \frac{\partial q_\Gamma}{\partial p^\Sigma}\frac{\partial^2 S}{\partial q_\Gamma \partial p^\Lambda}\right) = -2\left(\frac{\partial^2 S}{\partial q_\Lambda \partial q_\Sigma}\right)^{-1} \tag{2.23}$$

Note that the symmetry properties $F_{\Lambda\Sigma} = F_{\Sigma\Lambda}$ imply

$$\frac{\partial q_\Lambda}{\partial p^\Sigma} = \frac{\partial q_\Sigma}{\partial p^\Lambda} \tag{2.24}$$

$$\frac{\partial q_\Gamma}{\partial \phi^\Lambda}\frac{\partial^2 S}{\partial q_\Gamma \partial p^\Sigma} = \frac{\partial q_\Gamma}{\partial \phi^\Sigma}\frac{\partial^2 S}{\partial q_\Gamma \partial p^\Lambda} \tag{2.25}$$

$$\frac{\partial q_\Gamma}{\partial p^\Lambda}\frac{\partial^2 S}{\partial q_\Gamma \partial p^\Sigma} = \frac{\partial q_\Gamma}{\partial p^\Sigma}\frac{\partial^2 S}{\partial q_\Gamma \partial p^\Lambda} \tag{2.26}$$

which will help in the simplifications.

The change of variables in the differentials gives

$$\begin{aligned} dX^\Lambda \otimes d\bar{X}^\Sigma = &\left(\delta^\Lambda_\Gamma \delta^\Sigma_\Delta + \frac{\partial^2 S}{\partial q_\Lambda \partial p^\Gamma}\frac{\partial^2 S}{\partial q_\Sigma \partial p^\Delta} + i\left(\frac{\partial^2 S}{\partial q_\Lambda \partial p^\Gamma}\delta^\Sigma_\Delta - \frac{\partial^2 S}{\partial q_\Sigma \partial p^\Delta}\delta^\Lambda_\Gamma\right)\right) dp^\Gamma \otimes dp^\Delta \\ &+ \left(\frac{\partial^2 S}{\partial q_\Lambda \partial q_\Delta}\frac{\partial^2 S}{\partial q_\Sigma \partial p_\Gamma} + \frac{\partial^2 S}{\partial q_\Lambda \partial p_\Gamma}\frac{\partial^2 S}{\partial q_\Sigma \partial q_\Delta} + i\left(\frac{\partial^2 S}{\partial q_\Lambda \partial q_\Delta}\delta^\Sigma_\Gamma - \frac{\partial^2 S}{\partial q_\Sigma \partial q_\Delta}\delta^\Lambda_\Gamma\right)\right) dp^\Gamma \otimes dq_\Delta \\ &+ \left(\frac{\partial^2 S}{\partial q_\Lambda \partial q_\Gamma}\frac{\partial^2 S}{\partial q_\Sigma \partial q_\Delta}\right) dq_\Gamma \otimes dq_\Delta \end{aligned} \tag{2.27}$$

Finally, using again equations (2.12) to (2.16), and (2.24) to (2.26) the following expression is obtained:

$$g(p,q) = -2\left(\frac{\partial^2 S}{\partial p^\Lambda \partial p^\Sigma} dp^\Lambda \otimes dp^\Sigma + 2\frac{\partial^2 S}{\partial p^\Lambda \partial q_\Sigma} dp^\Lambda \otimes dq_\Sigma + \frac{\partial^2 S}{\partial q_\Lambda \partial q_\Sigma} dq_\Lambda \otimes dq_\Sigma\right) \tag{2.28}$$

Equation (2.28) implies that the Hessian matrix (1.7) is negative-definite.

Comparison of (2.28) with a different evaluation of the metric in Darboux coordinates (2.42) in section 2.4 , will allow us to prove that $M = -H$, as asserted in the introduction.

### 2.3 The Kähler form

In rigid special geometry the symplectic holomorphic vector $V = (X^\Lambda, F_\Lambda)$ defines the Kähler form through the formula [16]

$$\Omega_{IJ} dV^I \wedge d\bar{V}^J = -dX^\Lambda \wedge d\bar{F}_\Lambda + dF_\Lambda \wedge d\bar{X}^\Lambda \tag{2.29}$$

by writing $X^\Lambda = p^\Lambda + i\phi^\Lambda$ , $F_\Lambda = q_\Lambda + i\psi_\Lambda$, we find

$$-2dp^\Lambda \wedge dq_\Lambda - 2d\phi^\Lambda \wedge d\psi_\Lambda \tag{2.30}$$

On the other hand, if we compute (using the property $dX^\Lambda \wedge dF_\Lambda = 0$)

$$\begin{aligned} d(X^\Lambda + \bar{X}^\Lambda) \wedge d(F_\Lambda + \bar{F}_\Lambda) &= dX^\Lambda \wedge d\bar{F}_\Lambda + d\bar{X}^\Lambda \wedge dF_\Lambda \\ &= (dX^\Lambda \wedge d\bar{F}_\Lambda - dF_\Lambda \wedge d\bar{X}^\Lambda) = 4dp^\Lambda \wedge dq_\Lambda \end{aligned} \tag{2.31}$$

the following relation must hold,

$$dp^\Lambda \wedge dq_\Lambda = d\phi^\Lambda \wedge d\psi_\Lambda \tag{2.32}$$

We postpone in proving (2.32) but simply observe that

$$\omega = \frac{i}{4}\Omega_{IJ} dV^I \wedge d\bar{V}^J = \frac{i}{2}\Omega_{IJ} dP^I \wedge dP^J = i dq_\Lambda \wedge dp^\Lambda \tag{2.33}$$

### 2.4 The Kähler metric

We now consider the Kähler metric in complex coordinates. Let us first note a basic identity satisfied by the complex symplectic differential

$$dV = (dX^\Lambda, dF_\Lambda) = (dX^\Lambda, F_{\Lambda\Sigma} dX^\Sigma)$$

and the $M$ matrix defined in (1.6). It is easy to see that

$$M dV = i\Omega dV \tag{2.34}$$

and also to recall that $M$ is real, symmetric and symplectic, i.e. that it satisfies

$$M\Omega M = \Omega \tag{2.35}$$

The basic identity (2.34) allows us to prove that

$$
\begin{aligned}
d\bar{V}M(\mathcal{F})dV &= id\bar{V}\Omega dV = i\langle d\bar{V}, dV\rangle = i(d\bar{X}^\Lambda, d\bar{F}_\Lambda)\begin{pmatrix} -dF_\Lambda \\ dX^\Lambda \end{pmatrix} \\
&= i(dX^\Lambda d\bar{F}_\Lambda - d\bar{X}^\Lambda dF_\Lambda) = idX^\Lambda d\bar{X}^\Sigma(\bar{F}_{\Lambda\Sigma} - F_{\Lambda\Sigma}) = 2dX^\Lambda d\bar{X}^\Sigma \Im F_{\Lambda\Sigma}
\end{aligned}
\tag{2.36}
$$

Therefore the (positive-definite) Kähler metric is

$$
2\Im F_{\Lambda\Sigma} = \partial_\Lambda \bar{\partial}_\Sigma K \qquad K = -i\langle V, \bar{V}\rangle \tag{2.37}
$$

Let us now consider the Darboux (special) coordinates

$$
P^I = (\Re X^\Lambda, \Re F_\Lambda) = (p^\Lambda, q_\Lambda)
$$

and

$$
I^I = (\Im X^\Lambda, \Im F_\Lambda) = (\phi^\Lambda, \psi_\Lambda)
$$

such that

$$
dV^I = dP^I + idI^I \qquad d\bar{V}^I = dP^I - idI^I
$$

In components we have

$$
dV^I d\bar{V}^J = (dP^I + idI^I)(dP^J - idI^J) = (dP^I dP^J + dI^I dI^J) + i(dI^I dP^J - dP^I dI^J) \tag{2.38}
$$

$$
M_{IJ} dV^I d\bar{V}^J = M_{IJ}(dP^I dP^J + dI^I dI^J) \tag{2.39}
$$

We now compute $dI^I dI^J$ by using the following property

$$
I^I = \begin{pmatrix} \phi^\Lambda \\ \psi_\Lambda \end{pmatrix} = \begin{pmatrix} \frac{\partial S}{\partial q_\Lambda} \\ -\frac{\partial S}{\partial p^\Lambda} \end{pmatrix} = \Omega'^{IJ}\frac{\partial S}{\partial P^J} = -\Omega\frac{\partial S}{\partial P} \tag{2.40}
$$

where $\Omega'^{IJ} = -\Omega_{IJ}$ so that $\Omega'\Omega = \mathbf{1}$. It then follows that

$$
dI^I = \Omega' H dP \tag{2.41}
$$

where $H$ is the (real symmetric) Hessian of $S(P)$. By inserting (2.41) into (2.39) we obtain

$$
M_{IJ} dV^I \otimes d\bar{V}^J = (M - H\Omega M\Omega H)_{IJ} dP^I \otimes dP^J \tag{2.42}
$$

by comparing eq. (2.28) with (2.42) we get

$$
M_{IJ} dV^I \otimes d\bar{V}^J = -2H_{IJ} dP^I \otimes dP^J \tag{2.43}
$$

eqs. (2.42) and (2.43) then imply that $M = -H$. The same argument can be used for the Kähler form to prove equation (2.32). In fact

$$
\Omega_{IJ} dV^I \wedge d\bar{V}^J = \Omega_{IJ}(dP^I \wedge dP^J + dI^I \wedge dI^J) \tag{2.44}
$$

by use of (2.40), (2.41) we get

$$
\Omega_{IJ} dV^I \wedge d\bar{V}^J = dP^I \wedge dP^J (\Omega - M\Omega\Omega\Omega M)_{IJ} \tag{2.45}
$$

and, using $\Omega^3 = -\Omega$, $M\Omega M = \Omega$, we get

$$
\Omega_{IJ} dV^I \wedge d\bar{V}^J = 2\Omega_{IJ} dP^I \wedge dP^J \tag{2.46}
$$

which is nothing but equation (2.33)

### 2.5 The hyperkähler metric

The Darboux coordinates give also a striking simplification [17] of the metric of the real $4n$-manifold of the hypermultiplet geometry, which is obtained by the $c$-map introduced by [18] and which associates to any special Kähler manifold of dimension $n$ an hyperkähler manifold of complex dimension $2n$. Furthermore, as discussed in [26], this manifold has at least $2n$ isometries ($2n+3$ in the local case which come from the $c$-map construction [26]). The hypermultiplet geometry, as described in [18], has a metric of the form[2] [18, 26, 27]

$$g_{\Lambda\bar{\Sigma}} dz^\Lambda \otimes d\bar{z}^\Sigma + 2M_{IJ}(z,\bar{z}) dZ^I \otimes dZ^J \tag{2.47}$$

where $Z^I = (\zeta^\Lambda, \bar{\zeta}_\Lambda)$ are $2n$ real coordinates associated to a symplectic real vector $Z$ and $g_{\Lambda\bar{\Sigma}}$ is the original Kähler metric.

By adopting Darboux coordinates and noticing that $M = -H$ equation (2.46) takes the simple form

$$-2\frac{\partial^2 S}{\partial P^I \partial P^J}(dP^I \otimes dP^J + dZ^I \otimes dZ^J) = \frac{\partial^2 K}{\partial U^I \partial \bar{U}^J} dU^I \otimes d\bar{U}^J \tag{2.48}$$

It is immediate to see that (2.48) is a Kähler metric, for a complex $2n$-dimensional manifold with $2n$ complex coordinates given by

$$U^I = \frac{1}{2}(P^I + iZ^I) \tag{2.49}$$

and Kähler potential given by (1.16). Interestingly enough the hamiltonian function $S(P)$ has a double role. In the original symplectic manifold $\mathcal{M}$ of complex dimension $n$, its Hessian matrix in Darboux coordinates is the metric on the manifold. Considered as a function of "complex coordinates" $U^I$, it is the Kähler potential on the cotangent bundle (of the special manifold $\mathcal{M}$) with real dimension $4n$.

Note the obvious $2n$ isometries

$$U^I \longmapsto U^I + ia^I \tag{2.50}$$

as implied by the analysis of [26].

## 3. Central charges and special coordinate identities in complex and real coordinates

In the present section we discuss identities and relations of special geometry in presence of a background charge real symplectic vector $Q = (m^\Lambda, e_\Lambda)$. In terms of the special geometry complex sections the "central charge" function is given by

$$Z = \langle Q, V\rangle = Q^T \Omega V \tag{3.1}$$

[2] Note that because of property (2.35) if we lower the indices $Z_I$ the second term in (2.47) becomes $2(M^{-1})^{IJ} dZ_I \otimes dZ_J$. This agrees with the hypermultiplet metric as given in [19].



JHEP05(2006)008

where scalar symplectic products are understood. In local special geometry $Z$ is a section of a U(1) bundle over $\mathcal{M}$ and it is conveniently written in terms of symplectic sections over U(1) [24, 23, 8]

$$Z^L = \langle Q, V^L \rangle$$

$$V^L = (L^\Lambda, M_\Lambda) \qquad i\langle V^L, \bar{V}^L \rangle = 1 \tag{3.2}$$

so that

$$K = -\log(i(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda)) \tag{3.3}$$

In the rigid case the Kähler potential is rather given by

$$K = -i\langle V, \bar{V} \rangle = -i(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda) \tag{3.4}$$

so that

$$K_{\Lambda\bar{\Sigma}} = 2\Im F_{\Lambda\Sigma} > 0$$

Note that in the local case $\Im\mathcal{F}_{\Lambda\Sigma}$ is a matrix of lorentzian signature [23] with $n$ positive and one negative eigenvalues, while the matrix $\mathcal{N}$

$$\mathcal{N}_{\Lambda\Sigma} = \bar{h}_{I\Lambda}(\bar{f}^{-1})^I_\Sigma$$

(where $\bar{h}_{I\Lambda}$, $\bar{f}_{I\Lambda}$ are $(n+1)\times(n+1)$ matrices which are the components of the sections $\bar{D}_i\bar{V}$, $I = 1, \ldots, n$, and $V$, $I = 0$) is negative-definite (in the rigid case $\mathcal{N} \mapsto \bar{\mathcal{F}}$ so that $\Im\mathcal{N} \mapsto -\Im\mathcal{F}$, and $\Im\mathcal{F}$ becomes positive-definite and is an $n\times n$ instead of an $(n+1)\times(n+1)$ matrix).

In the local case, if we define the U(1) covariant derivatives $D_i Z^L$ ($\bar{D}_{\bar{i}} Z^L = 0$), $D_i V^L$ in terms of the U(1) sections $Z^L$ and $V^L$, then the following identity is true[3] [28]:

$$Q - i\Omega M(\mathcal{N})Q = -2i\bar{V}^L Z^L - 2ig^{i\bar{j}} D_i V^L \bar{D}_{\bar{j}} \bar{Z}^L \tag{3.5}$$

with the (local) special geometry identity

$$M(\mathcal{N})V^L = i\Omega V^L \tag{3.6}$$

where $M(\mathcal{N})$ is the same matrix as in (1.6) but with $\mathcal{F} \longmapsto \mathcal{N}$.

Note that if in (3.5) we take the scalar product with $Q$, since $\langle Q, Q \rangle = 0$ we obtain the black-hole potential as [8]

$$-\frac{1}{2} Q^T M(\mathcal{N}) Q = |Z^L|^2 + D_i Z^L \bar{D}_{\bar{j}} \bar{Z}^L g^{i\bar{j}} = V_{BH} \tag{3.7}$$

On the other hand, by multiplying by $V$ and using the fact that $MV = i\Omega V$, we obtain

$$\langle Q, V \rangle = Z$$

[3]Taking the real part of (3.5) we obtain the identity used in [29 – 31]. This identity has recently been generalized in the presence of more general fluxes in ref. [32].



JHEP05(2006)008

The rigid formula that replaces (3.5) is

$$Q + i\Omega M(\mathcal{F})Q = -2ig^{i\bar{j}}\partial_i V \bar{\partial}_{\bar{j}}\bar{Z} \tag{3.8}$$

which implies the rigid formula

$$V = \frac{1}{2}Q^T M(\mathcal{F})Q = g^{i\bar{j}}\partial_i Z \bar{\partial}_{\bar{j}}\bar{Z} \tag{3.9}$$

Note that (3.9), with respect to (3.7), loses the graviphoton charge contribution and it is identical to the $N = 1$ rigid formula for chiral multiplets of superpotential $Z$.

This formula coincides with the superpotential contribution to the $N = 2$ potential considered in ref. [33]. From (3.8) we also see that at a supersymmetric extremum $\partial_i Z = 0$ implies $Q = 0$, something which is different from the local case.

The local supersymmetric attractor point

$$D_i Z^L = 0$$

gives, instead, the so called "BPS attractor equations":

$$Q = -i(\bar{V}^L Z^L - V^L \bar{Z}^L) \tag{3.10}$$

$$\Omega M(\mathcal{N})Q = \bar{V}^L Z^L + V^L \bar{Z}^L \tag{3.11}$$

The rigid identities (3.8) can be written for real Darboux symplectic special coordinates by noticing that

$$Z = \langle Q, V\rangle = p^\Lambda e_\Lambda - m^\Lambda q_\Lambda + i(\phi^\Lambda - m^\Lambda \psi_\Lambda) = Z(Q, P, I) \tag{3.12}$$

By using now the property that

$$I = \Omega' \frac{\partial S}{\partial P}$$

we get

$$\frac{\partial Z}{\partial P^I} = -\Omega Q + iHQ = \tau Q \tag{3.13}$$

with $\tau = -\Omega + iH$, $\tau = -\tau^\dagger$.

Note that, as a consequence of the fact that $H\Omega H = \Omega$ we have

$$\tau\Omega\tau = 2\tau \qquad \tau\Omega\tau^T = 0$$

The potential becomes

$$V(P, Q) = -\frac{1}{2}Q^T HQ \tag{3.14}$$

### 3.1 Local special geometry

We finally discuss further identities of local special geometry. Since

$$V = (L^\Lambda, M_\Lambda = \mathcal{N}_{\Lambda\Sigma}L^\Sigma) \tag{3.15}$$



JHEP05(2006)008

and also

$$M_\Lambda = F_{\Lambda\Sigma} L^\Sigma \tag{3.16}$$

(whenever $F_\Lambda = \partial_\Lambda F$) it is also true that, in addition to (3.6), we have

$$M(\mathcal{F}) V^L = i\Omega V^L \tag{3.17}$$

by further use of the identity

$$dM_\Lambda = F_{\Lambda\Sigma\Delta} X^\Sigma dL^\Delta + F_{\Lambda\Sigma} dL^\Sigma = F_{\Lambda\Sigma} dL^\Sigma \tag{3.18}$$

(because $F_{\Lambda\Sigma\Delta} X^\Sigma = 0$) it is also true, as in the rigid case, that

$$M(\mathcal{F}) dV^L = i\Omega dV^L \tag{3.19}$$

By using the definition

$$D_i V^L = \partial V^L + \frac{1}{2} K_i V^L \tag{3.20}$$

we have

$$M(\mathcal{F}) D_i V^L = i\Omega D_i V^L \qquad (\bar{D}_{\bar{i}} V^L = 0) \tag{3.21}$$

where $D_i$ is a U(1) covariant derivative.

The Kähler metric on the manifold is given by [24]

$$g^L = i\langle \bar{D}\bar{V}^L, DV^L \rangle = i\bar{D}\bar{V}^L \Omega DV^L \tag{3.22}$$

By using the previous relations we get the equivalent expression

$$g^L = (\bar{D}\bar{V}^L M(\mathcal{F}) DV^L) \tag{3.23}$$

Formula (3.23) is the local analogue of (2.36) and it will be useful to formulate local special geometry in Darboux real special coordinates.

## Acknowledgments

We would like to thank M.A. Lledó for enlightening discussions.

S.F. would like to thank the Theoretical Physics Department at the University of Valencia where part of this work was performed. S.F. was partially supported by funds of the INFN-CICYT bilateral agreement. The work of S.F. has also been supported in part by the European Community Human Potential Program under contract MRTN-CT-2004-005104 “Constituents, fundamental forces and symmetries of the Universe”, in association with INFN Frascati National Laboratories and by the D.O.E. grant DE-FG03-91ER40662, Task C.

O.M. wants to thank the Department of Physics, Theory Division at CERN for its kind hospitality during the realization of this work. The work of O.M. has been supported by an FPI fellowship from the Spanish Ministerio de Educación y Ciencia (M.E.C.), through the grant FIS2005-02761 (M.E.C.) and EU FEDER funds, by the Generalitat Valenciana, contracts GV04B-226, GV05/102 and by the EU network MRTN-CT-2004-005104 “Constituents, fundamental forces and symmetries of the Universe”.

JHEP05(2006)008

## References


[1] B. de Wit and A. Van Proeyen, *Potentials and symmetries of general gauged $N = 2$ supergravity–Yang-Mills models*, *Nucl. Phys.* **B 245** (1984) 89.

[2] B. de Wit, P.G. Lauwers and A. Van Proeyen, *Lagrangians of $N = 2$ supergravity-matter systems*, *Nucl. Phys.* **B 255** (1985) 569.

[3] A. Strominger, *Special geometry*, *Commun. Math. Phys.* **133** (1990) 163.

[4] S. Ferrara and A. Strominger, *$N = 2$ space-time supersymmetry and Calabi-Yau moduli space*, presented at Strings '89 Workshop, College Station, Coll. Station Workshop (1989) 245;
P. Candelas and X. de la Ossa, *Moduli space of Calabi-Yau manifolds*, *Nucl. Phys.* **B 355** (1991) 455, Coll. Station Worksohp (1990) 401;
L. Castellani, R. D'Auria and S. Ferrara, *Special Kähler geometry: an intrinsic formulation from $N = 2$ space-time supersymmetry*, *Phys. Lett.* **B 241** (1990) 57.

[5] J. Polchinski and A. Strominger, *New vacua for type-II string theory*, *Phys. Lett.* **B 388** (1996) 736 [hep-th/9510227].

[6] T.R. Taylor and C. Vafa, *RR flux on Calabi-Yau and partial supersymmetry breaking*, *Phys. Lett.* **B 474** (2000) 130 [hep-th/9912152].

[7] S. Ferrara, R. Kallosh and A. Strominger, *$N = 2$ extremal black holes*, *Phys. Rev.* **D 52** (1995) 5412 [hep-th/9508072].

[8] S. Ferrara and R. Kallosh, *Supersymmetry and attractors*, *Phys. Rev.* **D 54** (1996) 1514 [hep-th/9602136].

[9] A. Strominger, *Macroscopic entropy of $N = 2$ extremal black holes*, *Phys. Lett.* **B 383** (1996) 39 [hep-th/9602111].

[10] G.W. Moore, *Arithmetic and attractors*, hep-th/9807087.

[11] G. Sierra and P.K. Townsend, *An introduction to $N = 2$ rigid supersymmetry*, in *Supersymmetry and supergravity 1983*, B. Milewski ed., World Scientific, Singapore 1983.

[12] S.J. Gates Jr., *Superspace formulation of new nonlinear sigma models*, *Nucl. Phys.* **B 238** (1984) 349.

[13] N. Seiberg, *Supersymmetry and nonperturbative beta functions*, *Phys. Lett.* **B 206** (1988) 75.

[14] L. Castellani, R. D'Auria and S. Ferrara, *Special Kähler geometry: an intrinsic formulation from $N = 2$ space-time supersymmetry*, *Phys. Lett.* **B 241** (1990) 57.

[15] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation and confinement in $N = 2$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 426** (1994) 19 [hep-th/9407087].

[16] D.S. Freed, *Special Kähler manifolds*, *Commun. Math. Phys.* **203** (1999) 31 [hep-th/9712042].

[17] N.G. Hitchin, *The moduli space of complex Lagrangian submanifolds*, *Asian J. Math.* **3** (1999) 77 [math.DG/9901069].

[18] S. Cecotti, S. Ferrara and L. Girardello, *Geometry of type-II superstrings and the moduli of superconformal field theories*, *Int. J. Mod. Phys.* **A 4** (1989) 2475.

[19] T. Mohaupt, *New developments in special geometry*, hep-th/0602171.


JHEP05(2006)008

[20] G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Black hole partition functions and duality*, *JHEP* **03** (2006) 074 [hep-th/0601108].

[21] H. Ooguri, A. Strominger and C. Vafa, *Black hole attractors and the topological string*, *Phys. Rev.* **D 70** (2004) 106007 [hep-th/0405146].

[22] L. Andrianopoli et al., $N = 2$ *supergravity and* $N = 2$ *super Yang-Mills theory on general scalar manifolds: symplectic covariance, gaugings and the momentum map*, *J. Geom. Phys.* **23** (1997) 111 [hep-th/9605032].

[23] A. Ceresole, R. D'Auria and S. Ferrara, *The symplectic structure of* $N = 2$ *supergravity and its central extension*, *Nucl. Phys.* **46** *(Proc. Suppl.)* (1996) 67 [hep-th/9509160].

[24] A. Ceresole, R. D'Auria, S. Ferrara and A. Van Proeyen, *Duality transformations in supersymmetric Yang-Mills theories coupled to supergravity*, *Nucl. Phys.* **B 444** (1995) 92 [hep-th/9502072].

[25] A. Ceresole, R. D'Auria and S. Ferrara, *On the geometry of moduli space of vacua in* $N = 2$ *supersymmetric Yang-Mills theory*, *Phys. Lett.* **B 339** (1994) 71 [hep-th/9408036].

[26] S. Ferrara and S. Sabharwal, *Quaternionic manifolds for type-II superstring vacua of Calabi-Yau spaces*, *Nucl. Phys.* **B 332** (1990) 317.

[27] R. D'Auria, S. Ferrara, M. Trigiante and S. Vaula, *Gauging the Heisenberg algebra of special quaternionic manifolds*, *Phys. Lett.* **B 610** (2005) 147 [hep-th/0410290].

[28] S. Bellucci, S. Ferrara and A. Marrani, *On some properties of the attractor equations*, *Phys. Lett.* **B 635** (2006) 172 [hep-th/0602161].

[29] S. Ferrara, M. Bodner and A.C. Cadavid, *Calabi-Yau supermoduli space, field strength duality and mirror manifolds*, *Phys. Lett.* **B 247** (1990) 25.

[30] R. Kallosh, *New attractors*, *JHEP* **12** (2005) 022 [hep-th/0510024].

[31] R. Kallosh, N. Sivanandam and M. Soroush, *The non-BPS black hole attractor equation*, *JHEP* **03** (2006) 060 [hep-th/0602005].

[32] G. Dall'Agata, *Non-Kähler attracting manifolds*, hep-th/0602045.

[33] I. Antoniadis, H. Partouche and T.R. Taylor, *Spontaneous breaking of* $N = 2$ *global supersymmetry*, *Phys. Lett.* **B 372** (1996) 83 [hep-th/9512006].

JHEP05(2006)008